Chapter 5
Analysis of Multiple Well Systems

5-1. General Equations

In most applications, a system of pressure relief wells in various arrays is required for the relief of substratum pressures or reduction of ground-water levels. In such cases, analyses must be made to determine the number and spacing of wells to meet these requirements. The head at any point p produced by a system of fully penetrating artesian wells was first determined by Forcheimer (1914). His general equation as later modified by Dachler (1936) is

$$h_p = H_1 - \frac{1}{2\pi kD}\left(Q_{w1} \ln \frac{R_1}{r_1} + Q_{w2} \ln \frac{R_2}{r_2} + \ldots\ldots Q_{wn} \ln \frac{R_n}{r_n}\right) \tag{5-1}$$

or

$$h_p = H_1 - \frac{1}{2\pi kD}\left(\sum_{i=1}^{i=n} Q_{wi} \ln \frac{R_i}{r_i}\right) \tag{5-2}$$

where

H = gross head on system

n = number of wells in group

Q_{wi} = discharge from ith well

R_i = radius of influence of ith well

r_i = distance from ith well to point at which head is computed

The head, h_{wj}, at any well, e.g. well j, in a system of n wells is determined from the equation

$$h_{wj} = H_1 - \frac{1}{2\pi kD}\left(Q_{wj} \ln \frac{R_j}{r_{wj}} + \sum_{i=1}^{i=n-1} Q_{wi} \ln \frac{R_i}{r_{i,j}}\right) \tag{5-3}$$

where

Q_{wj} = flow from well j

R_j = radius of influence of well j

r_{wj} = effective well radius of well j

$r_{i,j}$ = distance from each well to well j

The other symbols are as defined previously. Equations 5-1 and 5-3 as well as subsequent equations for multiple well systems are based on the principle of superposition. Thus, the head at a given well in a system of wells is equal to that resulting from this well flowing as if no other wells were present minus the head reduction caused at the well due to flow from the remaining wells. In most applications, the radius of influence is large compared to the distance between wells and can be considered as constant. When wells are pumped as in a dewatering system, the values of Q_{wi} are known (or assumed). However, when n wells are used for pressure relief where they flow under artesian head conditions, the flow from each well must be computed taking into account the discharge elevation of each well. The procedure requires the solution of n simultaneous equations to determine individual well flows.

5-2. Empirical Method

An empirical method developed by Warriner and Banks (1977) using the results of electrical analogy studies by Duncan (1963) and Banks (1965) can be used to determine the head at any point within a random array of fully or partially penetrating wells. The method, described in EM 1110-2-1901, is also valid for arbitrarily shaped source boundaries. A FORTRAN computer code is provided by Warriner and Banks (1977).

5-3. Circular Source

a. General case. The general equations for a group of fully penetrating wells subject to seepage from a circular source with radius *R* are shown in Figure 5-1. It is assumed that the radius *R* is large with respect to the distances between wells and that the flows from each well are equal. As indicated previously in the case of variable well discharges, the procedure requires the solution of *n* simultaneous to solve for individual well flows.

b. Circular array of wells. A special case consists of a circular array of *n* wells equally spaced along the circumference of a circle of radius r_c, the center of which is also the center of a circular source of seepage of radius *R*. The general equations are shown in Figure 5-2.

c. Other well arrays. For other multiple-well systems within a circular source, see Muskat (1937), Banks (1963), and TM 5-818-5.

5-4. Wells Adjacent to Infinite Line Source with Impervious Top Stratum

Where wells are located adjacent to a source which can be approximated as an infinite line source and the pervious stratum is overlain by an impervious top stratum extending landward to a great distance, a solution for heads and well flows is obtained using the method of images. The equations are shown in Figure 5-3 for the case of (a) equal well discharges and (b) variable well discharges. As noted previously, case (b) requires the solution of *n* simultaneous equations to determine individual well flows.

5-5. Infinite Line of Wells

An infinite line of wells refers to a system of wells that conforms approximately to the following idealized conditions:

a. The wells are equally spaced and identical in dimensions.

b. The pervious stratum is of uniform depth and permeability along the entire length of the system.

c. The effective source of flow and the effective landside exit or block, if present, are parallel to the line of the wells.

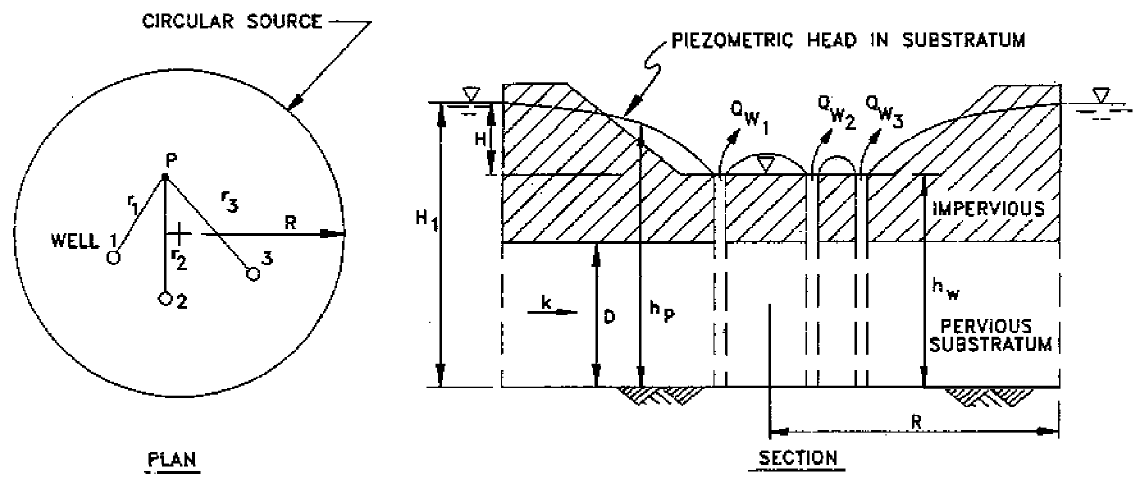
d. The boundaries at the ends of the system are impervious, normal to the line of the wells, and at a distance equal to one-half the well spacing beyond the end of the well system. For the above conditions, the flow to each well and the pressure distribution around each well are uniform for all wells along the line. Therefore, there is no flow across planes centered between wells and normal to the line, hence no overall longitudinal component of the flow exists anywhere in the system. The term infinite is applied to such a system because it may be analyzed mathematically by considering an infinite number of wells; the actual number of wells in the system may be from one to infinity.

5-6. Top Stratum Conditions

The permeability and lateral extent of the top stratum landward of an infinite line of wells can have a pronounced effect on the performance of the well system. The assumption of a completely impervious top stratum extending landward to infinity is a convenient assumption for which theoretical solutions are available. However, this condition is rarely realized in practice. A more general condition occurs when the impervious top stratum extends landward a finite distance terminating at a line sink. This condition is also applicable with respect to results at the well line to the case of a semipervious top stratum which can be converted to an equivalent length of impervious top stratum using appropriate blanket formulas. The two conditions are illustrated in Figure 5-4 together with assumed head distributions with and without relief wells including the effects of well losses. Calculation of the corrected net head on the well system, *h*, should also take into consideration any extension of the well riser above tailwater elevation. A third condition occurs when the pervious substratum is blocked at some point landward of the well line. Theoretical solutions for the three conditions follow.

5-7. Infinite Line of Wells, Impervious Top Stratum

The head midway between wells and the well flows for the case of an impervious top stratum extending landward a great distance ($L_3 = \infty$) may be calculated using the method of multiple images (after Muskat 1937, Middlebrooks and Jervis 1947). Solutions are shown in Figure 5-5 for the case of no well losses. Equations 5-14 through 5-17 are applicable to both fully penetrating and partially penetrating wells. The latter make use of the so called well factors, Θ_a and Θ_m.



The head at Point P is:

$$h_p = H_1 - \frac{1}{2\pi kD}\left(Q_{w1} \ln \frac{R}{r_1} + Q_{w2} \ln \frac{R}{r_2} + \ldots\ldots Q_{wn} \ln \frac{R}{r_n}\right) \tag{5-4}$$

or

$$h_p = H_1 - \frac{1}{2\pi kD} \sum_{i=1}^{i=n} Q_{wi} \ln \frac{R}{r_i} \tag{5-5}$$

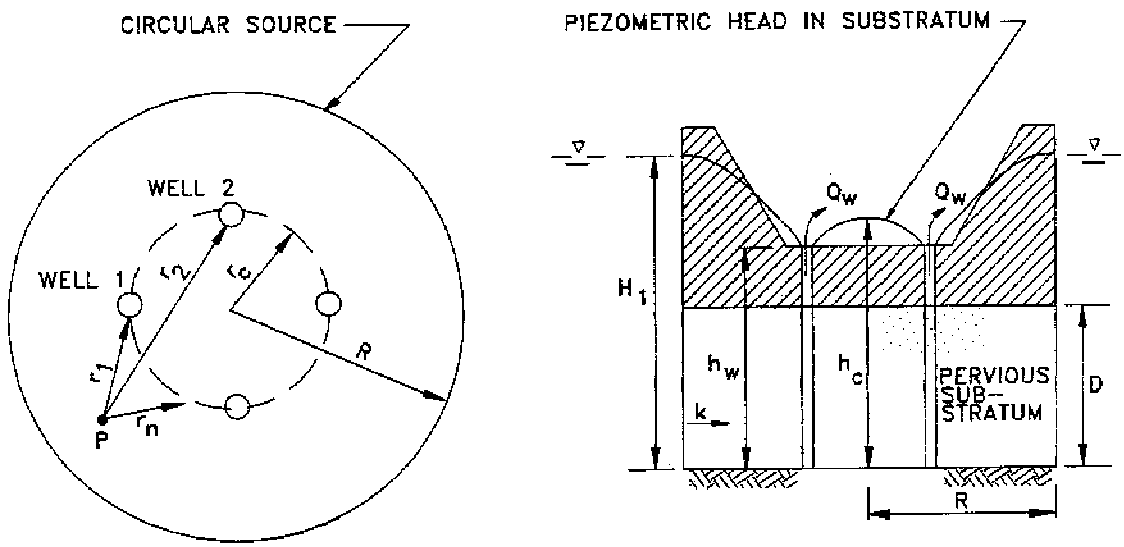
The head at Well 1 is:

$$h_{w1} = H_1 - \frac{1}{2\pi rD}\left(Q_{w1} \ln \frac{r}{r_{w1}} + \sum_{i=2}^{i=n} Q_{wi} \ln \frac{R}{r_i}\right) \tag{5-6}$$

If all wells have the same radius, and discharge at the same elevation, h_w, then the well discharges are equal and given by:

$$Q_w = \frac{2\pi kD\,(H_1 - h_w)}{n \sum_{n=1}^{n=n} \ln \frac{R}{r_w}} \tag{5-7}$$

Figure 5-1. Random array of fully penetrating wells with a circular source



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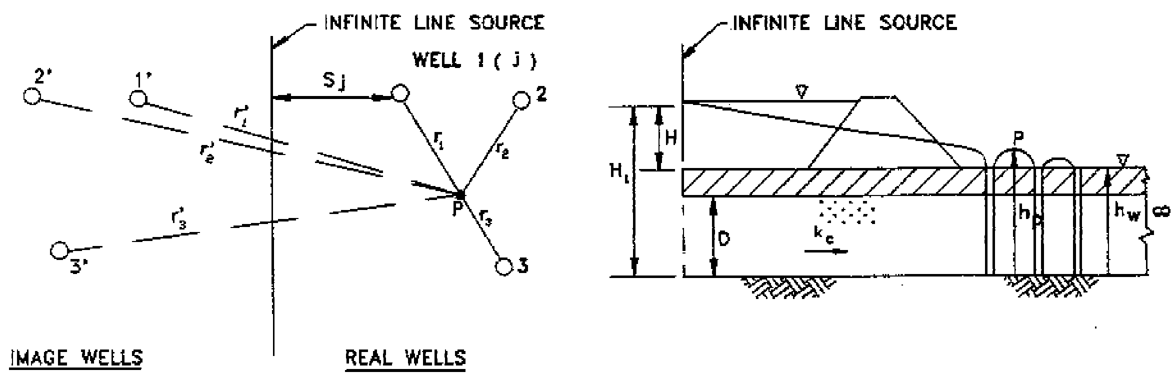
$$h_p = H_1 - \frac{nQ_w}{2\pi kD}\left(\ln R - \sum_{m=1}^{m=n} \ln r_m\right) \tag{5-8}$$

$$h_c = H_1 - \frac{nQ_w}{2\pi kD} \ln \frac{R}{r_c} \tag{5-9}$$

$$Q_w = \frac{2\pi kD\,(H_1 - h_w)}{\ln \dfrac{R^n}{r_c^{n-1} r_w} - \sum\limits_{m=1}^{m=n-1} \ln 2 \sin \dfrac{m\pi}{n}} \tag{5-10}$$

n = number of wells

Figure 5-2. Circular array of fully penetrating artesian wells with a circular source



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A. For equal well discharges

$$h_p = H_1 - \frac{Q_w}{2\pi kD} \sum_{i=1}^{i=n} \ln \frac{r_i'}{r_i} \tag{5-11}$$

$$Q_w = \frac{2\pi kD\,(H - h_w)}{\ln \dfrac{2S_j}{r_w} + \sum\limits_{i=1}^{i=n, i\neq j} \ln \dfrac{r_j'}{r_j}} \tag{5-12}$$

B. For variable well discharges

$$h_p = H_1 - \frac{1}{2\pi kD} \sum_{i=1}^{i=n} Q_{wi} \ln \frac{r_i'}{r_i} \tag{5-13}$$

where
r_i = distance from Point P to real Well i
r_i' = distance from Point P to image Well i
r_j = distance from Well j to real Well i
r_j' = distance from Well j to image Well i

Figure 5-3. Multiple wells adjacent to infinite line source - general case

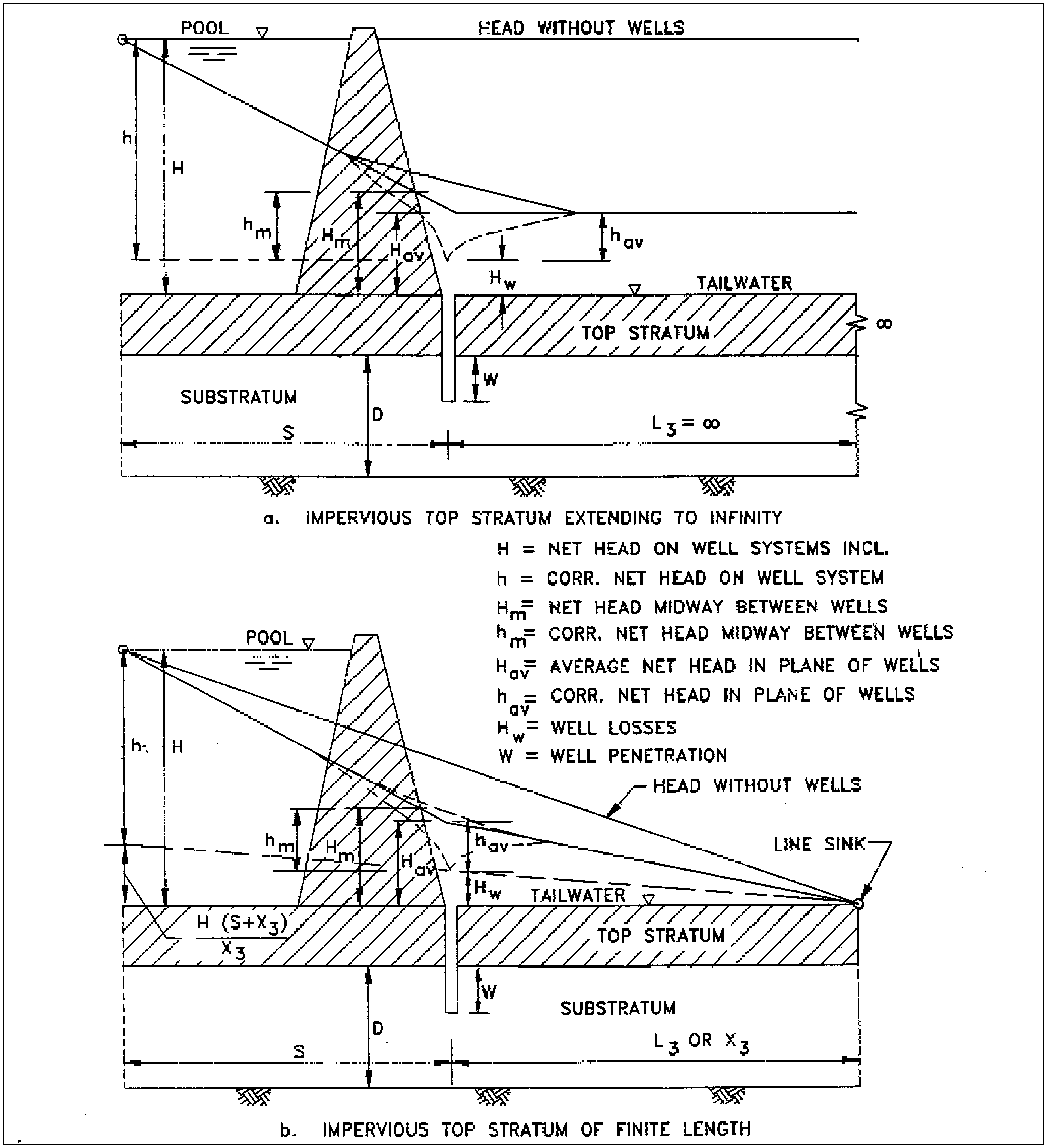
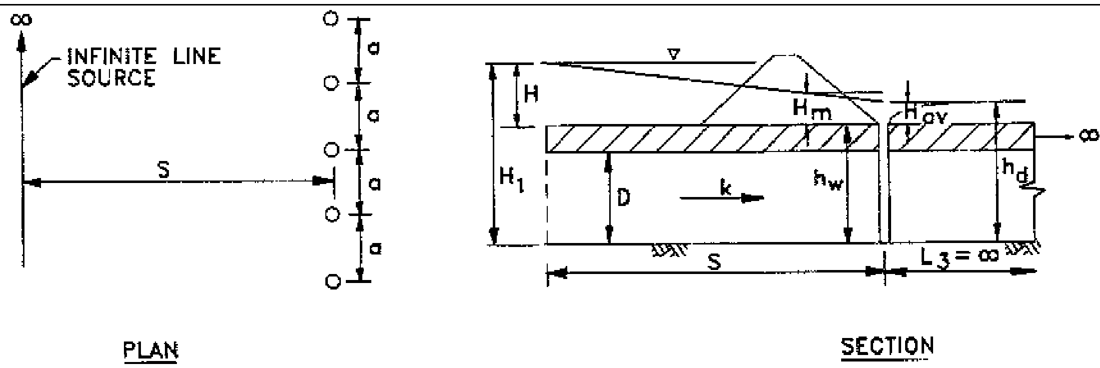


Figure 5-4. Infinite line of wells with infinite or finite impervious top stratum - general case



$$H_m = \frac{(H_1 - h_w)\,\theta_m}{\frac{S}{a} + \theta_a} = \frac{H\theta_m}{\frac{S}{a} + \theta_a} \tag{5-14}$$

$$H_{av} = \frac{(H_1 - h_w)\,\theta_a}{\frac{S}{a} + \theta_a} = \frac{H\theta_a}{\frac{S}{a} + \theta_a} \tag{5-15}$$

$$h_d = h_w + H_{av} \tag{5-16}$$

$$Q_w = \frac{(H_i - h_w)\,kD}{\frac{S}{a} + \theta_a} = \frac{kDH_a}{S + a\theta_a} \tag{5-17}$$

Note: Equations are valid for both fully and partially penetrating well systems. Effects of well losses ignored.

Figure 5-5. Infinite line of wells parallel to infinite line source - impervious top stratum

5-8. Well Factors

The well factor, Θ_a, is the "extra length" or average uplift factor, and Θ_m is the midwell uplift factor. For fully penetrating wells,

$$\theta_a = \frac{1}{2\pi} \ln \frac{a}{2\pi r_w} \tag{5-18}$$

$$\theta_m = \frac{1}{2\pi} \ln \frac{a}{\pi r_w} \tag{5-19}$$

Approximate solutions for the well factors for various well penetrations were developed by Bennett and Barron (1957). More theoretically exact solutions were developed by Barron (1982) and verified by electrical analogy tests. The theoretical results are shown in Table 5-1 and plotted in Figures 5-6 and 5-7 together with the data from the electrical analogy tests. As there is a linear relation between the well factors and log a/r_w for values of a/r_w greater than about 20, the well factors are shown in terms of values at a/r_w = 100. The well factors at any other value of a/r_w are given by the following equations:

$$\theta_a = \theta_a\ (a/r_w = 100) + \Delta\theta(\log a/r_w - 2) \tag{5-20}$$

$$\theta_m = \theta_m(a/r_w = 100) + \Delta\theta(\log a/r_w - 2) \tag{5-21}$$

where $\Delta\Theta$ is obtained from Table 5-1. Values of the well factors may also be obtained from the nomograph from EM 1110-2-1901 shown in Figure 5-8 (after Bennett and Barron 1957). The nomograph though based on approximate solutions, is reasonably accurate for well penetrations greater than 25 percent. A computer program for well design based on the Figure 5-8 was developed by Conroy (1984).

5-9. Infinite Line of Wells, Impervious Top Stratum of Finite Length

In many instances, the impervious top stratum landward of a line of wells is of finite length, and the boundary edge can be considered as a line sink. The presence of exposed borrow pits or other seepage exits landward of the well line can be simulated by a line sink. The head distribution beneath the top stratum without wells varies linearly from 100 percent of the net head at the effective source of seepage to 0 percent at the line sink. The conditions are illustrated in Figure 5-4 (b). These conditions are also applicable to the case of a semipervious landside blanket after conversion to an equivalent length of impervious blanket x_3 . Equations for the head midway between wells and well flows are shown in Figure 5-9. The equations are applicable to both fully penetrating and partially penetrating well systems. The equations in Figure 5-9 apply to the case of no well losses. If well losses are considered, substitute h for H as shown in Figure 5-4 (b).

5-10. Infinite Line of Wells, Impervious Top Stratum Extending to Blocked Exit

Pervious foundations seldom extend landward to a great distance. Blockades generally occur because of the presence of old clay-filled channels or upland formations. If the distance from the line of wells is large, then the approximation of an infinite landward extent is reasonable. If the distance from the line of wells is less than the well spacing, then the error due to the approximation may be significant. The equations for the head midway between wells and well flow are shown in Figure 5-10 with exact equations for the case of fully penetrating wells and reasonably accurate equations for both fully and partially penetrating wells where the distance to the blocked exit is greater than one-half times the well spacing. The presence of a blocked exit can be ignored if the equivalent length of landside impervious top stratum is less than L_B .

5-11. Infinite Line of Wells, Discharge Below Ground Surface

In many well installations, the well outlets are located below the ground surface to prevent any seepage upward through the top stratum. Under this condition, the blanket formulas are inapplicable and the top stratum is assumed to be impervious. Solutions are obtained using equations in Figure 5-5, with h_d at or below ground surface, assuming $\Delta h_d = H_{av}$.

5-12. Infinite Line of Wells, No Top Stratum

A special case may exist in which there is no landside top stratum and wells are needed to lower the heads below the landward ground surface. The flow in this case is a combination of artesian and gravity flow, and

Table 5-1
Theoretical Values of Θ_a and Θ_m

W/D	D/a	a/r_w	Θ_a	Θ_m	$\Delta\Theta$
100%	All values	100	0.440	0.550	1.00
75%	0.25	100	0.523	0.633	0.489
	0.50		0.563	0.667	
	1.0		0.606	0.681	
	2.0		0.678	0.682	
	3.0		0.748	0.682	
	4.0		0.818	0.682	
50%	0.25	100	0.742	0.851	0.733
	0.40		0.857	0.955	
	1.0		0.983	1.012	
	2.0		1.175	1.024	
	3.0		1.361	1.024	
	4.0		1.547	1.024	
25%	0.25	100	1.225	1.335	1.466
	0.50		1.569	1.622	
	1.0		1.926	1.908	
	2.0		2.390	2.024	
	3.0		2.798	2.047	
	4.0		3.199	2.075	
15%	0.25	100	1.662	1.772	2.077
	0.50		2.310	2.401	
	1.0		2.970	2.938	
	2.0		3.747	3.293	
	4.0		4.941	3.432	
10%	0.25	100	1.908	2.018	3.298
	0.50		2.934	3.025	
	1.0		3.977	3.941	
	2.0		5.139	4.649	
	4.0		6.814	5.071	
5%	0.25	100	1.778	1.887	6.963
	0.50		3.879	3.969	
	1.0		6.063	6.021	
	2.0		8.377	7.864	
	4.0		11.144	9.283	

the equations shown in Figure 5-11 (Johnson 1947) may be used to estimate heads midway between wells and well flows for design.

5-13. Finite Well Lines, Infinite Line Source

The essential difference between finite and infinite well lines is the presence or absence of an appreciable component of flow parallel to the line of wells, resulting in nonuniform distribution of heads midway between wells and well discharges.

a. Impervious top stratum. Where the landside top stratum is impervious and extends landward to infinity, the solution for a linear array of equispaced wells parallel to an infinite line source can be obtained using the equations shown in Figure 5-3.

b. Impervious top stratum of finite length. In the case of an impervious top stratum extending to a finite distance landward of the well line or in the case of a semipervious landside top stratum converted to an equivalent length of impervious top stratum, theoretical solutions for finite well lines are not available. Empirical solutions based on electrical analogy tests are presented in EM 1110-2-1905. The application of these solutions for design is discussed in Chapter 7.

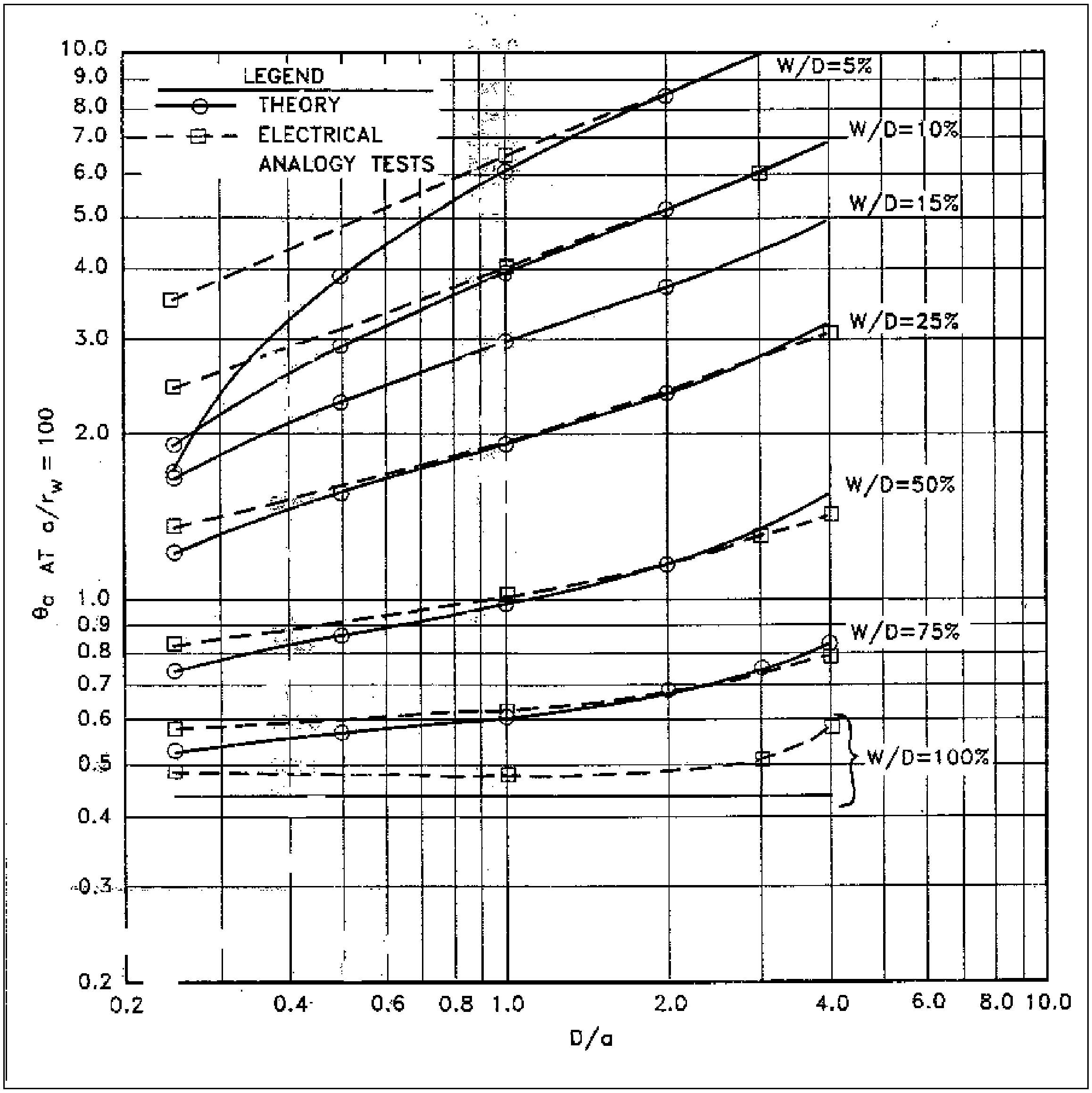


Figure 5-6. Theoretical values of average uplift factor (after Barron 1982)

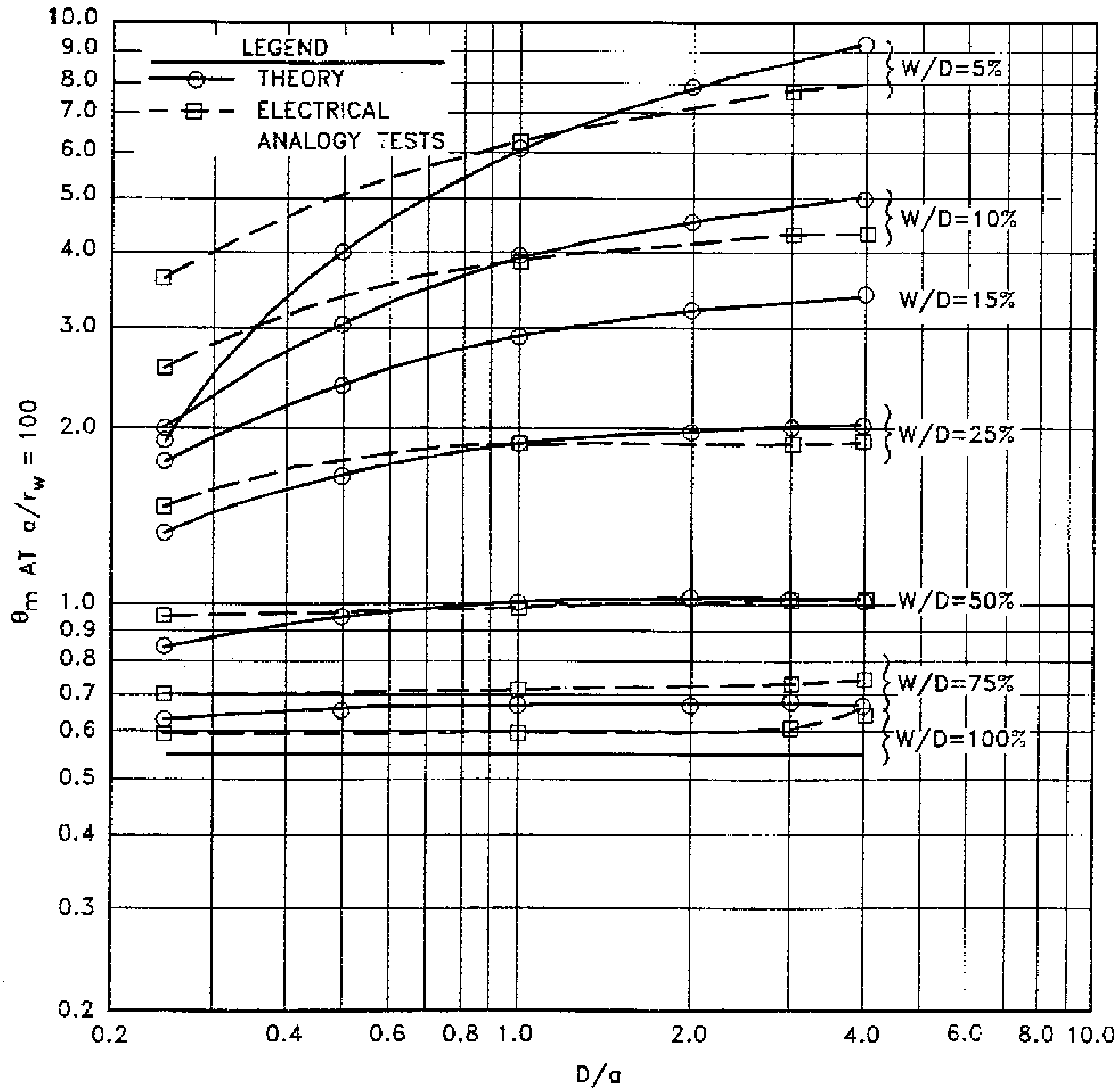


Figure 5-7. Theoretical values of midwell uplift factor (after Barron 1982)

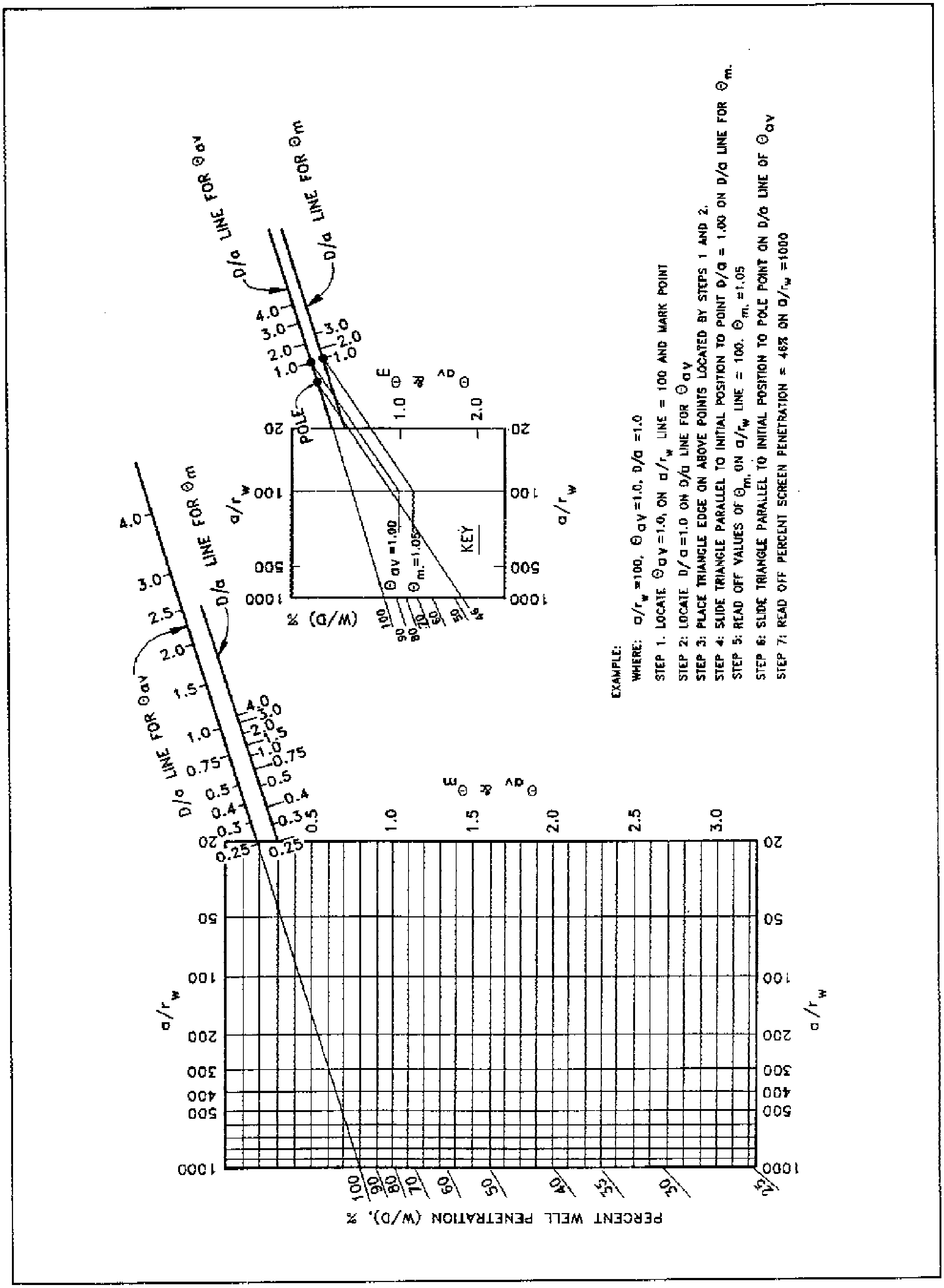
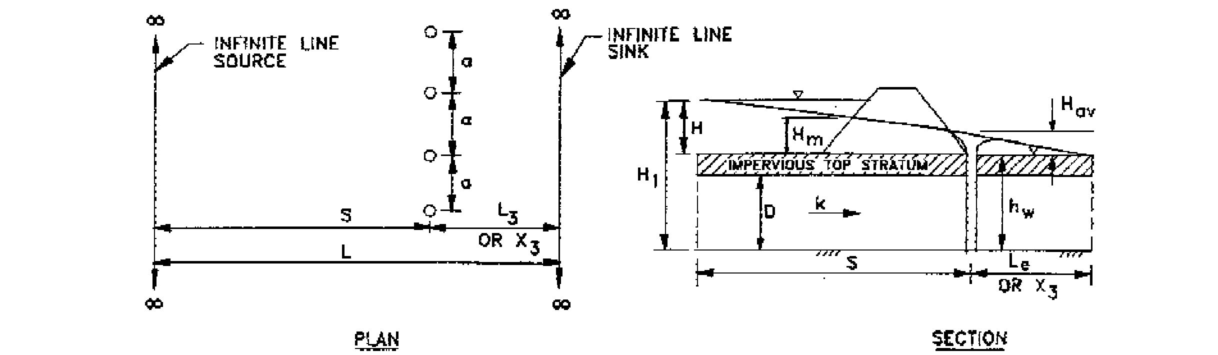


Figure 5-8. Nomographic chart for design of relief well systems



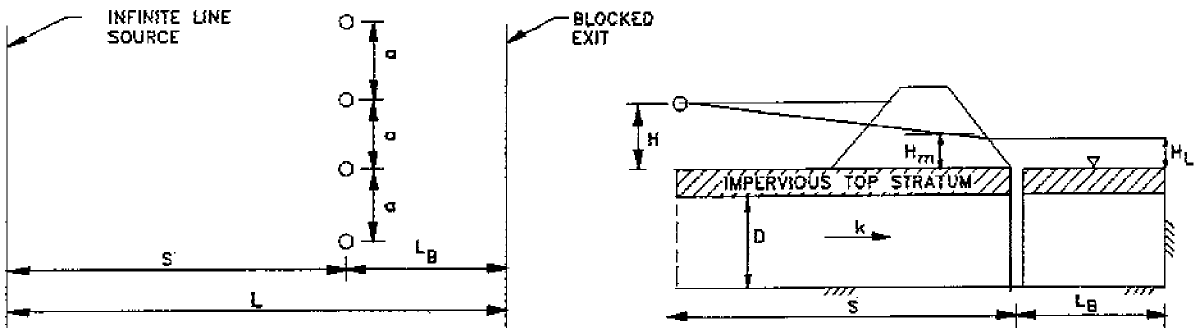
$$H_m = \frac{(H_1 - h_w)\,\theta_m}{\frac{S}{a} + \left(\frac{L}{L_e}\right)\theta_a} = \frac{H\,\theta_m}{\frac{S}{a} + \left(\frac{L}{L_e}\right)\theta_a} \tag{5-22}$$

$$H_{av} = \frac{(H_1 - h_w)\,\theta_a}{\frac{S}{a} + \left(\frac{L}{L_e}\right)\theta_a} = \frac{H\,\theta_a}{\frac{S}{a} + \left(\frac{L}{L_e}\right)\theta_a} \tag{5-23}$$

$$Q_w = \frac{(H_1 - h_w)\,k\,D}{\frac{S}{a} + \left(\frac{L}{L_e}\right)\theta_a} = \frac{H\,k\,D}{\frac{S}{a} + \left(\frac{L}{L_e}\right)\theta_a} \tag{5-24}$$

Note: Equations are valid for both fully and partially penetrating well systems. Effects of well losses ignored.

Figure 5-9. Infinite line of wells parallel to infinite line source, impervious top stratum of finite length



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Fully penetrating wells

$$H_m = h - \frac{Q_w}{2\pi kD}\,\frac{1}{\displaystyle\sum_1^\infty \ln \frac{\left\{\left[2\tan\frac{\pi S}{2L}\cosh^2\frac{\pi a}{4L}\left(n-\frac{1}{2}\right)\right]^2 + \sinh^2\frac{\pi}{2L}\left(n-\frac{1}{2}\right)\right\}}{\left\{\left[2\tan\frac{\pi S}{2L}\cosh^2\frac{4a}{\pi L}\left(n-\frac{1}{2}\right)\right]^2 + \sinh^2\frac{\pi}{2L}\left(n-\frac{1}{2}\right)\right\}}} \qquad (5\text{-}25)$$

$$H_L = H - \frac{Q_w}{2\pi kD}\sum_1^\infty \ln\left[\frac{\cosh\frac{\pi a}{2L}\left(n-\frac{1}{2}\right)+\sin\frac{\pi s}{2L}}{\cosh\frac{\pi a}{2L}\left(n-\frac{1}{2}\right)-\sin\frac{\pi S}{2L}}\right] \qquad (5\text{-}26)$$

$$Q_w = \frac{1}{\ln\frac{4L}{\pi r_w}\tan\frac{\pi S}{2L} + \displaystyle\sum_1^\infty \ln\left[\frac{\left(2\tan\frac{\pi S}{2L}\cosh^2\frac{\pi na}{4L}\right)^2+\sinh^2\frac{\pi na}{2L}}{\left(2\tan\frac{\pi S}{2L}\sinh^2\frac{\pi na}{4L}\right)^2+\sinh^2\frac{\pi na}{2L}}\right]} \qquad (5\text{-}27)$$

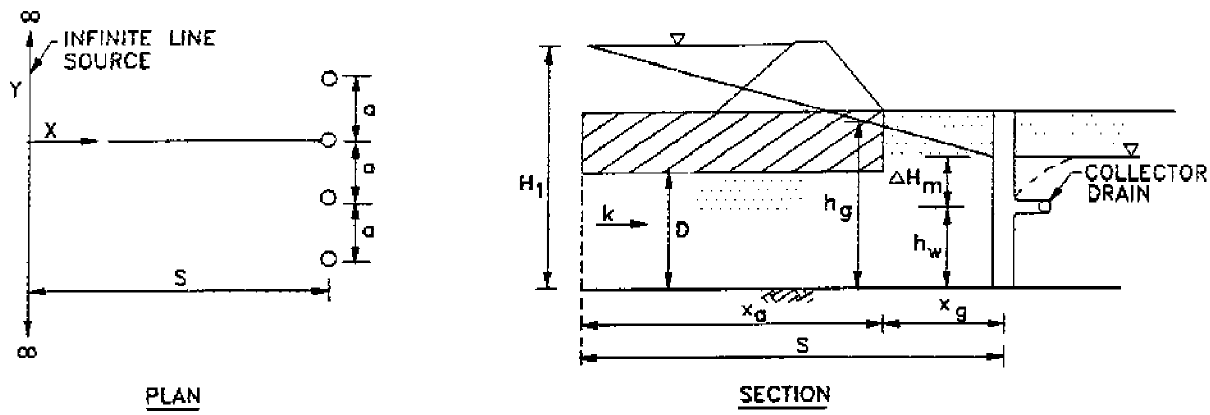
For partially penetrating wells and $\frac{L_B}{a} > 0.5$

$$H_m = \frac{Q_w\theta_m}{kD} \qquad (5\text{-}28)$$

$$H_L = \frac{Q_w\theta_a}{kD} \qquad (5\text{-}29)$$

$$Q_w = \frac{kDH}{\frac{S}{a}+\theta_a} \qquad (5\text{-}30)$$

Figure 5-10. Infinite line of wells with blocked landside exit



Gravity flow, Q_g = artesian flow Q_a

$$Q_g = \pi k \frac{\left(h_g^2 - h_w^2\right)}{\ln \dfrac{a e^{\frac{2\pi x_g}{a}}}{2\pi r_w}} = Q_a = \frac{k\,(H_1 - h_g)\,aD}{x_a} \tag{5-31}$$

$$h_g = -\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \beta H_1 + h_w^2} \tag{5-32}$$

$$\text{where} \quad \beta = \frac{aD}{\pi x_a} \ln \left(\frac{a e^{\frac{2\pi x_g}{a}}}{2\pi r_w} \right) \tag{5-33}$$

h_p = height of saturation at any point in gravity flow zone

$$h_p = \sqrt{h_g^2 - \frac{\left(h_g^2 - h_w^2\right)}{\ln \dfrac{a e^{\frac{2\pi x_a}{a}}}{2\pi r_w}} \ln \left[\frac{\cosh \frac{2\pi}{a}(x + x_a) - \cos \frac{2\pi y}{a}}{\cosh \frac{2\pi}{a}(x - x_a) - \cos \frac{2\pi y}{a}} \right]} \tag{5-34}$$

ΔH_m = excess head above the well outlet midway between wells

$$\Delta H_m = \sqrt{h_g^2 - \frac{\left(h_g^2 - h_w^2\right)}{\ln \dfrac{a e^{\frac{2\pi x_a}{a}}}{2\pi r_w}} \ln \frac{e^{\frac{2\pi x_a}{a}}}{2}} - h_w \tag{5-35}$$

Figure 5-11. Infinite line of fully penetrating wells, combined gravity, and artesian flow